J. Fluid Mech. (1988), *vol.* 186, *pp.* 337–349
Printed in Great Britain

Capillary–gravity waves produced by a heaving body

By L. M. HOCKING
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

(Received 20 November 1986 and in revised form 18 June 1987)

The vertical oscillation of a partially submerged body produces a surface wave that carries energy away from the body. The amplitude of this wave, when surface-tension effects are not negligible, depends on the conditions applied at the line of contact between the body and the free surface of the fluid. An edge condition that includes both dynamic variation of the contact angle and contact-angle hysteresis is used in this paper; the condition implies a dissipation of energy at the contact line. The amplitude of the wave and the amount of energy dissipated are calculated for a horizontal circular cylinder and for a simple source-and-plate model. This model is shown to be an adequate representation for the qualitative description of heaving motions and to simplify the calculations considerably. The effects of varying the relative importance of surface tension and gravity, the dynamic behaviour of the contact angle, and the amount of hysteresis, are calculated for the source-and-plate model.

1. Introduction

The heaving motion of a partially immersed body displaces the fluid surrounding it. Waves are thereby generated on the free surface of the fluid which carry energy away from the body, resulting in the decay of a free motion of the body or requiring an input of energy to sustain a forced motion. This problem has been the topic of many studies, but for gravity waves only. The case of a horizontal circular cylinder with its axis in the free surface was solved by Ursell (1949) and the heaving motion of a vertical cylinder has been studied by Yeung (1981). Little attention has so far been given to the modifying effects of capillarity, since the main application has been to ship motions and capillarity is quite unimportant in such cases. However, on the scale of laboratory experiments it may not always be justifiable to neglect surface tension and it is of some interest to determine the radiation of energy from a heaving body by capillary–gravity waves.

The presence of surface tension adds a term to the dynamic boundary condition at the free surface that expresses the normal stress balance there. For propagating waves in regions where there are no boundaries that cross the free surface, the sole effect of surface tension is to change the dispersion relation, at least for small-amplitude waves. When boundaries that intersect the free surface are present, it is necessary, in order to close the formulation of the problem, to impose a condition at the line of contact between the free surface and the boundary. The choice of an appropriate condition is not immediately apparent, yet different conditions may produce dramatically different motions. It is important to include in the edge condition terms that model two phenomena that are known to characterize contact angles at moving contact lines. The first of them is known as contact-angle hysteresis. A range of static contact angles is possible and motion of the contact line only occurs when the angle is sufficiently large or sufficiently small, depending on the direction of motion of the contact line. The second phenomenon is dynamic contact-angle variation; when the contact line is moving the contact angle varies with its velocity. An important feature of a model that incorporates these effects is that energy can be dissipated at the contact line; this dissipation is the result of microscale physical processes taking place in the vicinity of the contact line whose effect on macroscale phenomena is reflected in the contact-angle behaviour. There is no dissipation in the special cases of a fixed contact line and of a fixed contact angle.

A model edge condition permitting realistic contact-angle behaviour has been used in some previous investigations. The motion of the contact line on a vertically oscillating plate has been considered by Young and Davis (1987), in a parameter range for which the motion of the contact line could be uncoupled from the fluid motion. A different parameter range for the same system was examined by Hocking (1987*b*) and the radiated waves were determined. In this problem the edge condition is the sole cause of the fluid motion; viscous effects do not appear at leading order and fluid is not displaced by the plate. If, however, the plate is replaced by a body with a non-zero immersed volume, both the displacement of the fluid by the body and the edge condition will contribute to the generation of surface waves. It is then natural to enquire into the relative importance of these two causes of the radiation of energy away from the body. The amount of energy dissipated may be reduced substantially by dissipation at the contact line; for the vertically oscillating plate, such dissipation can account for as much as 90% of the energy input.

An edge condition that includes contact-angle hysteresis introduces an element of nonlinearity into the otherwise linear problem of small-amplitude waves. It is helpful, therefore, to simplify the problem as much as possible, while retaining the main features of the motion. It does not seem likely that the shape of the immersed portion of the body will of great significance. The width of the body at the waterline is important, however, since this controls the volume of fluid displaced during the heaving motion. A simplification that includes both displacement and edge effects is afforded by replacing the body by a vertical plate and a submerged oscillatory source. Then the plate induces waves through the edge condition but not by displacing fluid, whereas the opposite is true for the source. The strength of the source can be adjusted to give the same displacement of fluid as a body of given width, and varying the depth of the source will model a one-parameter family of body shapes. We consider first the waves produced by a body whose submerged cross-section is a semicircle and then the source-and-plate model, both without hysteresis. Comparison of the two sets of results provides some evidence of the value of the model. The solution for the source and plate is obtained with contact-angle hysteresis included. In all cases, the main quantities of interest are the amplitudes of the radiated waves and the proportions of the energy input that are radiated and dissipated.

Only two-dimensional motions are considered, so the radiated waves are plane. It is also assumed that the body surface is vertical where it intersects the free surface and that the static contact angle is 90° to ensure a flat interface in equilibrium. The amplitude of the motion is assumed to be sufficiently small for a linearized solution to be valid. In order that hysteresis effects may be included with small-amplitude waves present, it is necessary to restrict the static range of contact angles to be small also. For a fixed hysteresis range, the contact line would remain fixed throughout the motion of the wave amplitude were small enough. The properties of capillary–gravity waves with fixed edges were examined by Benjamin & Scott (1979) and by Graham-Eagle (1984), who initiated the recent study of capillary–gravity waves in the presence of boundaries. Evans (1968) had earlier considered the reflection of capillary–gravity waves by a barrier and the radiation of waves from a partially submerged cylinder. He recognized the importance of the edge condition but assumed that the slope of the free surface at the edge had a harmonic oscillation with a prescribed amplitude. He made no attempt to relate this amplitude with the wave motion, although it clearly cannot be chosen independently, and his solution therefore contains an arbitrary parameter. Rhodes-Robinson (1971) made a similar assumption in his treatment of the waves produced by a vertical wavemaker moving horizontally.

In addition to the dissipation resulting from the edge condition there is also a loss of energy because of viscous stresses. The relative importance of these two damping mechanisms is discussed in Hocking (1987*a*) for a standing-wave problem, in which the edge condition is incorporated. The largest viscous effect comes from the oscillatory Stokes layer on the body and has a magnitude proportional to $Re^{-\frac{1}{2}}$ (Re is the Reynolds number). This viscous correction is not included in the present work, in which viscosity is ignored throughout.

2. Formulation

With the x-axis horizontal and the y-axis along the upward vertical from an origin in the free surface, the mean position of the submerged portion of the body can be defined by

$$y = -D(x), \tag{2.1}$$

with D an even function of x, $D(b) = 0$ and $D'(b) = \infty$. The body is forced to move vertically with velocity V sin $\bar{\sigma}\bar{t}$. All lengths are scaled by $1/\bar{k}$, where $2\pi/\bar{k}$ is the wavelength of surface waves of angular frequency $\bar{\sigma}$, so that

$$\bar{\sigma}^2 = g\bar{k} + \frac{\gamma\bar{k}^3}{\rho}, \tag{2.2}$$

where g is gravity, γ surface tension and ρ the density of the fluid. Velocities are scaled by V, and $\boldsymbol{u} = (u, v)$, p and t are the non-dimensional fluid velocity, the dynamic pressure and the time respectively. The equations for the linearized fluid motion can then be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\operatorname{grad} p, \quad \operatorname{div} \boldsymbol{u} = 0, \tag{2.3}$$

subject to the conditions

$$|\boldsymbol{u}| \rightarrow 0 \quad \text{as } y \rightarrow -\infty, \tag{2.4}$$

$$\boldsymbol{n}\cdot(\boldsymbol{u} - \sin \sigma t \boldsymbol{j}) = 0 \quad \text{on } y = -D(x), \quad |x| < b, \tag{2.5}$$

$$\frac{\partial \eta}{\partial t} = v, \quad \eta - K\frac{\partial^2 \eta}{\partial x^2} = p \quad \text{on } y = 0, \quad |x| > b, \tag{2.6}$$

where $\boldsymbol{n}$ is the outward normal to the body surface, $\boldsymbol{j}$ is the unit vector in the y-direction, η is the free-surface elevation, and

$$K = \frac{\gamma\bar{k}^2}{\rho g}, \quad \sigma^2 = 1 + K. \tag{2.7}$$

Since $D(x)$ is even, we need only consider the region $x \geqslant 0$. The edge condition at $x = b$ has the form

$$\frac{\partial \eta}{\partial t} - \sin \sigma t = \begin{cases} \lambda\left(\dfrac{\partial \eta}{\partial x} - \alpha\right) & \text{for } \left|\dfrac{\partial \eta}{\partial x}\right| > \alpha, \\ 0 & \text{for } \left|\dfrac{\partial \eta}{\partial x}\right| < \alpha, \\ \lambda\left(\dfrac{\partial \eta}{\partial x} + \alpha\right) & \text{for } \dfrac{\partial \eta}{\partial x} < -\alpha, \end{cases} \tag{2.8}$$

where λ and α are constants determining the dynamic variation of the contact angle and the amount of hysteresis respectively. For simplicity, the advancing and retreating values have been chosen symmetrically.

The energy balance over one period and for unit span in the half-space $x > 0$ can be written in the form

$$E_{\mathrm{S}} = E_{\mathrm{R}} + E_{\mathrm{D}}, \tag{2.9}$$

where the suffixes refer to the supplied, radiated and dissipated energies. Following the evaluation given in Hocking (1987 *b*) and omitting contributions to the energy flux that have zero mean over a period, we can express these energies in the forms

$$E_{\mathrm{S}} = \int_0^{2\pi/\sigma} \left[-K \sin \sigma t \frac{\partial \eta}{\partial x} + \int p(\boldsymbol{u} \cdot \boldsymbol{n})\, \mathrm{d}s \right] \mathrm{d}t, \tag{2.10}$$

$$E_{\mathrm{D}} = \int_0^{2\pi/\sigma} K\left(\frac{\partial \eta}{\partial t} - \sin \sigma t\right) \frac{\partial \eta}{\partial x}\, \mathrm{d}t, \tag{2.11}$$

$$E_{\mathrm{R}} = \tfrac{1}{2}\pi(1+3K)\,|C|^2, \tag{2.12}$$

where $\partial \eta / \partial x$ and $\partial \eta / \partial t$ are evaluated at $x = b$, the integral in the arclength s is taken along the body surface and $|C|$ is the amplitude of the radiated wave of frequency σ.

For the source-and-plate model, the edge condition (2.8) is to be applied at $x = 0$, and the normal velocity condition (2.5) is now $u = 0$ on the plate, so that u vanishes on $x = 0$ for all negative values of y and the depth of the bottom of the plate is irrelevant. There is a source of strength $-2b \sin \sigma t$ at $x = 0$, $y = -d$, which has the same displacement effect as that of a body of width $2b$. The energy-flux balance produces the same integrals as before (2.10) and (2.11), except that the derivatives of η are now to be evaluated at $x = 0$ and the integral along the body surface is now an integral round a semicircle of small radius centred at the source.

3. Semicircular cylinder without hysteresis

For a cylinder whose submerged portion has a semicircular cross-section,

$$D(x) = (b^2 - x^2)^{\frac{1}{2}}. \tag{3.1}$$

We can work in terms of the pressure, which is a harmonic function, and a solution that satisfies the normal velocity condition (2.5) is given by the real part of

$$p_{\mathrm{s}} = \sigma b^2 \exp(\mathrm{i}\sigma t) \frac{y}{(x^2 + y^2)}. \tag{3.2}$$

If we write $p = p_s + q\exp(i\sigma t)$, we then have to seek the harmonic function $q(x, y)$, which is to be an even function of x and to have zero normal derivative on the semicircle. If we define two functions $f(x)$ and $g(x)$ for $x > b$ by

$$f(x) = \frac{\partial q(x,0)}{\partial y}, \quad g(x) = q(x,0), \tag{3.3}$$

the free-surface conditions (2.6) satisfied by $\eta\exp(i\sigma t)$ take the forms

$$\eta = \frac{b^2}{\sigma x^2} + \frac{f(x)}{\sigma^2} \tag{3.4}$$

and

$$\eta - K\frac{d^2\eta}{dx^2} = g(x). \tag{3.5}$$

This equation can be solved for η subject to the condition that η be bounded at infinity, and when we equate this value of η to that given by (3.4) we obtain the equation

$$\tfrac{1}{2}K^{-\frac{1}{2}}\int_b^\infty \exp\left(-K^{-\frac{1}{2}}|x_1 - x|\right) g(x_1)\,dx_1 - A\,K^{\frac{1}{2}}\exp\{-K^{-\frac{1}{2}}(x-b)\} = \frac{b^2}{\sigma x^2} + \frac{f(x)}{\sigma^2}, \tag{3.6}$$

where A is an arbitrary constant. The edge condition (2.8) with $\alpha = 0$ can then be written in the form

$$i\frac{f(b)}{\sigma} + 2i = \lambda B, \tag{3.7}$$

where

$$B = A + \tfrac{1}{2}K^{-\frac{1}{2}}\int_b^\infty \exp\{K^{-\frac{1}{2}}(b - x_1)\}\,g(x_1)\,dx_1; \tag{3.8}$$

B is the value of $d\eta/dx$ at $x = b$.

The solution can be obtained if we can find a relation connecting $f(x)$ and $g(x)$. To do this, we map the region occupied by the fluid into the lower half-plane by

$$X + iY = x + iy + \frac{b^2}{(x+iy)}, \tag{3.9}$$

so that the semicircle becomes $Y = 0$, $|X| < 2b$ and the free surface is $Y = 0$, $|X| > 2b$. If $p(x,y) \equiv P(X, Y)$, a source distribution along $Y = 0$ gives the equation

$$P(X,0) = -\pi^{-1}\int_{2b}^\infty \log(X_1^2 - X^2)\,\frac{\partial P(X_1,0)}{\partial Y}\,dX_1, \tag{3.10}$$

since $\partial P/\partial Y$ vanishes on $Y = 0$ for $|X| < 2b$ and P is an even function of X. Reverting to the original coordinates, we can write this equation in the form

$$g(x) = -\pi^{-1}\int_b^\infty f(x_1)\log\left\{(x_1^2 - x^2)\left(1 - \frac{b^4}{x_1^2 x^2}\right)\right\}dx_1. \tag{3.11}$$

We have to solve the pair of integral equations (3.6) and (3.11) and the edge condition (3.7) to determine f, g and B or A. An alternative form for (3.7) can be found if we use (3.4) to determine B. Instead of (3.7), we have the condition

$$f(b) + 2\sigma = -i\lambda\left(\frac{f'(b)}{\sigma} - \frac{2}{b}\right). \tag{3.12}$$

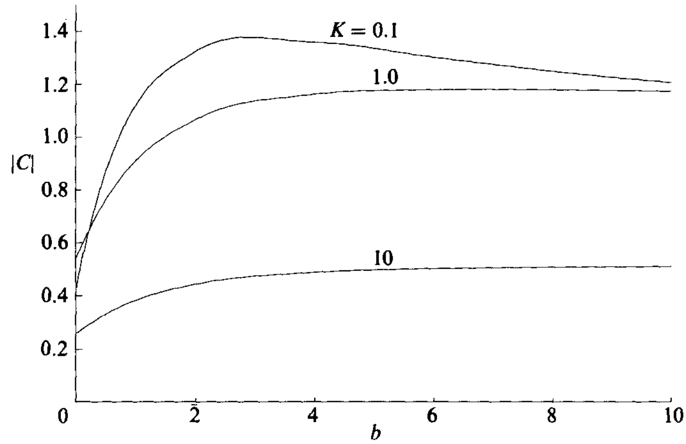

FIGURE 1. Wave amplitude for an oscillating horizontal cylinder. The contact line is fixed on the cylinder.

For large values of x, we have outgoing waves of amplitude C if

$$\eta \sim C \exp\{-\mathrm{i}(x-b)\}$$

and the asymptotic value of q is given by

$$q \sim \sigma^2 C \exp\{y-\mathrm{i}(x-b)\}. \tag{3.13}$$

When the corresponding asymptotic values of f and g are subtracted, the remainders are to be $o(1)$ at infinity, a condition that eliminates any incoming wave.

The calculation of the remaining parts of f and g and of the constants B and C was done by using a finite-difference approximation to reduce the integral equations to two sets of linear algebraic equations. The pair of integral equations could be reduced analytically to a single equation, but it is very complicated and of no computational advantage. Some care had to be exercised to deal adequately with the truncation of the infinite integrals and with the logarithmic kernel in (3.11). An analytical evaluation was used for the integral in (3.6) from the last mesh point to infinity. Integration by parts was used to overcome the corresponding difficulty of an infinite range of the integral in (3.11). The singularity of the kernel in (3.11) at $x_1 = x$ was allowed for by an exact integration across the singularity, assuming a linear variation of $f(x_1)$ there. Fuller details of the numerical method employed can be found in Hocking (1987c), in which a similar pair of integral equations had to be solved.

Some results of these calculations are shown in figures 1 and 2. In figure 1 the amplitude of the outgoing wave is plotted as a function of the radius of the body for three values of K and for a fixed contact line. This is an important special case, since it occurs when there is large hysteresis present. For $b = 0$, it is the edge condition only that produces waves, while for other values of b the displacement effect is also present. Since the edge effect does not depend directly on b, the figure shows that the displacement effect is of the same order of magnitude. Figure 2 displays the supplied and radiated energies as functions of λ, for three values of K and of b. The dissipated energy is represented by the gap between the two curves. The energy supplied decreases monotonically as λ increases, since less energy is needed to oscillate the

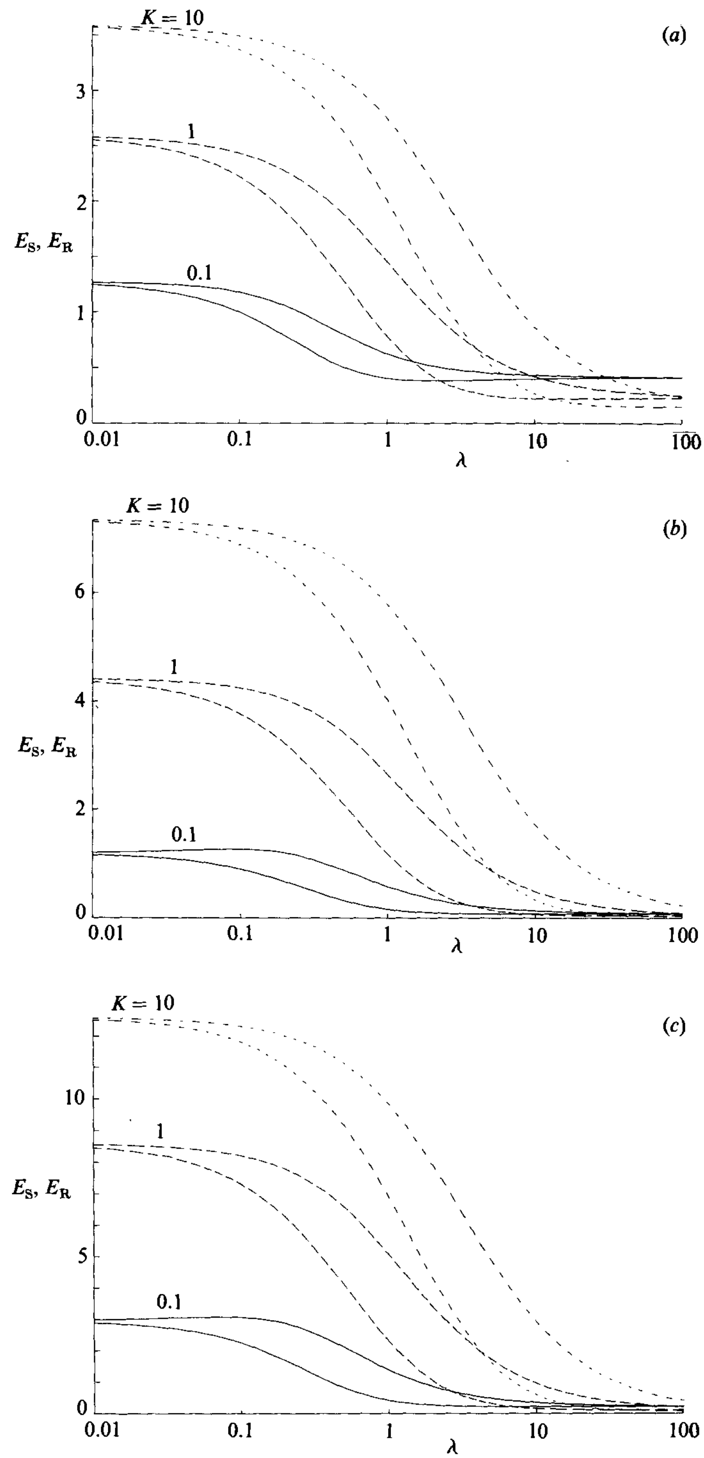

FIGURE 2. Input and radiated energies for a horizontal cylinder. The distance between the curves represents the dissipated energy. (*a*) $b = 0.1$, (*b*) 1.0, (*c*) 10.0.

body when the contact line can slip on the body surface than when it is fixed there. The curves for different values of b are similar to each other in shape, but the magnitudes of the energies increase with b.

The calculations of Ursell (1949) for pure gravity waves predated the development of computers. His method was to expand the potential as a series of functions that satisfied the free-surface condition and to apply collocation on the cylinder to satisfy the boundary condition there. A severely truncated series was used and the coefficients were determined by a least-squares estimate. His results, in the present notation, correspond to $K = 0$, $\lambda = \infty$, $0 < b < 5$. It is not clear whether Ursell's method could be adapted to the capillary–gravity wave problem because of the changed form of the free-surface condition. On the other hand, the results for zero surface tension ($K = 0$) cannot be obtained by the method used in this paper, since zero K is a singular limit of the equations. However, the free-surface condition (3.5) suggests that, for small K, the wave amplitude should have a factor $(1-cK^{\frac{1}{2}})$ for some constant c (possibly depending on the value of b). The smallest value of K used in the present calculations is $K = 0.1$, and if the ratio of the results to those of Ursell is calculated, the constant c turns out to have a value of very nearly 0.6 for all the values of b tested.

4. Source and plate, without hysteresis

For a plate lying along $x = 0$ and a source of strength $-2b \sin \sigma t$ at $(0, -d)$, the solution can be found more simply than for a cylinder since no mapping of the fluid region is required. The analysis follows lines very similar to those used in Hocking (1987*b*). The pressure can be written as the real part of $p \exp(\mathrm{i}\sigma t)$ and, as in §3, we can write $p = p_s + q$, where

$$p_s = \tfrac{1}{2} b\sigma\pi^{-1}[\log\{x^2+(y+d)^2\} - \log\{x^2+(y-d)^2\}] \tag{4.1}$$

and

$$q = \int_0^\infty P(k) \cos kx \exp(ky)\,\mathrm{d}k. \tag{4.2}$$

The chosen value of p_s accounts for the source at $(0, -d)$ and an image source at $(0, d)$, and a cosine transform for q can be used since q is an even function of x.

The free-surface conditions (2.6) become

$$\eta = \frac{2bd}{\pi\sigma(x^2+d^2)} + \sigma^{-2}\int_0^\infty kP \cos kx\,\mathrm{d}k, \tag{4.3}$$

$$\eta - K\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} = \int_0^\infty P \cos kx\,\mathrm{d}k. \tag{4.4}$$

Following the procedure explained in Hocking (1978*b*), we find that

$$P = -\frac{2\sigma\{\sigma KB + b(1+Kk^2)\,\mathrm{e}^{-kd}\}}{\pi\{k(1+Kk^2)-\sigma^2\}} + \sigma^2 C\delta(k-1), \tag{4.5}$$

where B is the slope of the free surface at the contact line and C is a constant. The asymptotic value of η as $x \rightarrow \infty$ then shows that

$$\eta \sim C \exp(-\mathrm{i}x), \tag{4.6}$$

so that we have an outgoing wave of amplitude C, provided that

$$C = \frac{2\mathrm{i}\{KB+\sigma b \exp(-d)\}}{1+3K}. \tag{4.7}$$

The value of η at the contact line can be found by setting $x = 0$ in (4.3) and the edge condition then determines the value of B and hence of C. The result is that

$$B = \frac{1+3K-2\sigma^2\pi^{-1}b(J_2-\mathrm{i}\pi\mathrm{e}^{-d})}{2\sigma\pi^{-1}KJ_1-\mathrm{i}\{\lambda(1+3K)+2\sigma K\}}, \tag{4.8}$$

where

$$J_1 = \int_0^\infty \frac{(1+3K)\,k}{(k-1)(Kk^2+Kk+K+1)}\mathrm{d}k$$

$$= \tfrac{1}{2}\log\left(\frac{K+1}{K}\right)+\frac{3K+2}{K^{\frac{1}{2}}(3K+4)^{\frac{1}{2}}}\tan^{-1}\left(\frac{3K+4}{K}\right)^{\frac{1}{2}}, \tag{4.9}$$

$$J_2 = \int_0^\infty \frac{(1+3K)\,\mathrm{e}^{-kd}}{(k-1)(Kk^2+Kk+K+1)}\mathrm{d}k$$

$$= -\mathrm{e}^{-d}\mathrm{Ei}(d)-\int_0^\infty \frac{K(k+2)\,\mathrm{e}^{-kd}}{Kk^2+Kk+K+1}\mathrm{d}k. \tag{4.10}$$

The exponential integral is defined by

$$\mathrm{Ei}(d) = \Gamma+\log d+\sum_{n=1}^{\infty}\frac{d^n}{n\,n!}, \tag{4.11}$$

where Γ is Euler's constant.

The amplitude of the wave is given by (4.7) and the values of the energies can be found from (2.10)–(2.12). In terms of B_{r} and B_{i}, the real and imaginary parts of B, which can be calculated from (4.8), we find that

$$|C| = 2K(1+3K)^{-1}\{(B_{\mathrm{r}}+\sigma bK^{-1}\mathrm{e}^{-d})^2+B_{\mathrm{i}}^2\}^{\frac{1}{2}}, \tag{4.12}$$

$$E_{\mathrm{R}} = \tfrac{1}{2}\pi(1+3K)\,|C|^2, \tag{4.13}$$

$$E_{\mathrm{D}} = \frac{\lambda\pi K\,|B|^2}{\sigma}, \tag{4.14}$$

$$E_{\mathrm{S}} = E_{\mathrm{R}}+E_{\mathrm{D}}. \tag{4.15}$$

The calculation of these quantities is easily performed, since only a quadrature is required. There are four parameters, λ, K, b and d, and these are too many to permit more than a very incomplete presentation of the results. Since the model was proposed as a replacement for more general body shapes, the results presented are those that enable a comparison to be made with those for the semicircular cylinder. It is expected and confirmed that appropriate values of the depth of the source, d, are roughly comparable with the radius of the semicircle. Figures 3 and 4 are for $d = b$ and show the quantities corresponding to those presented in figures 1 and 2 (b).

A comparison of figures 1 and 3 indicates a similar dependence of the two sets of results on K and b, except that for $K = 0.1$ the wave amplitude shows a more rapid decrease from its peak value as b is increased for the source and plate than for the cylinder. The curves in figures 2 (b) and 4 are nearly identical, showing that the

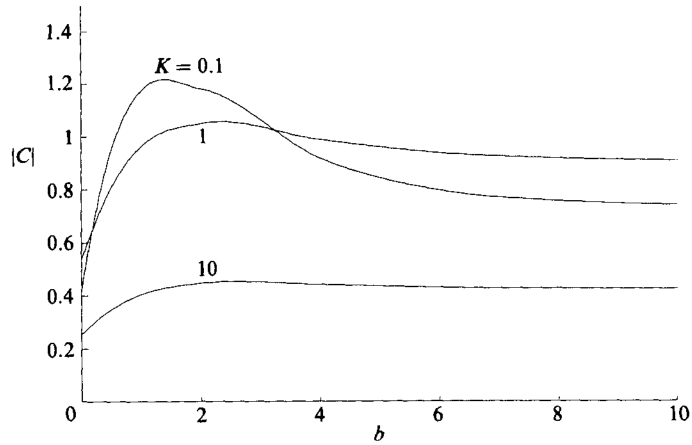

FIGURE 3. Wave amplitude for a source and plate, with fixed contact line: $b = d = 1$.

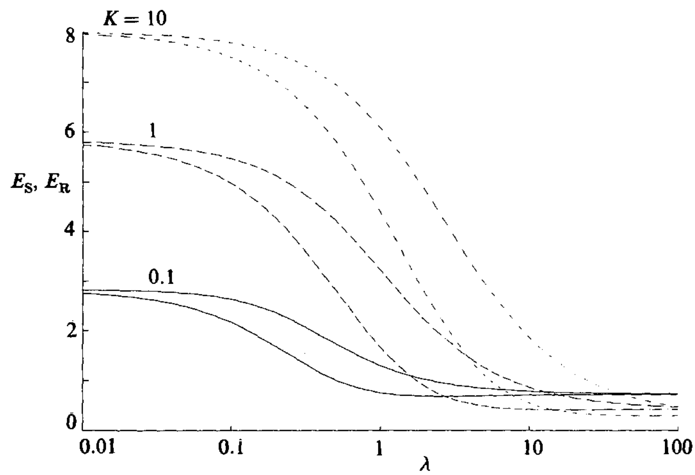

FIGURE 4. Input and radiated energies for a source and plate: $b = d = 1$.

results in the two cases are qualitatively similar (note the different scales of the ordinates). This correspondence between the two sets of results provides evidence of the value of the source-and-plate model as a simple replacement for bodies of general shapes when their heaving motion is to be investigated. Consequently, we can now proceed, with this justification, to examine the simple model with hysteresis of the contact angle included.

5. Source and plate, with hysteresis

When hysteresis is included, the changing boundary condition at the edge during the course of the motion prevents the solution from being sinusoidal. An appropriate procedure is to take a Laplace transform of the solution, starting from rest. With a similar notation to that used in §4, we obtain, instead of (4.5), the result that

$$\bar{P} = -\frac{2b}{\pi}\frac{\sigma s}{s^2+\sigma^2}\frac{(1+Kk^2)\,\mathrm{e}^{-kd}}{s^2+k(1+Kk^2)}+\frac{2K}{\pi}\frac{s\bar{B}}{s^2+k(1+Kk^2)}, \tag{5.1}$$

where the overbar denotes the Laplace transform and s is the parameter of the transform. The value of η at the contact line can then be calculated and the edge condition (2.8), after inverting the transforms, gives an integral equation for $B(t)$, the slope of the free surface at the contact line. This equation can be written in the form

$$-\int_0^t F(\tau)\,B'(t-\tau)\,\mathrm{d}\tau - \sin\sigma t + 2b\sigma\pi^{-1}H(t) = \begin{cases} \lambda\{B(t)-\alpha\} & \text{for } B > \alpha, \\ 0 & \text{for } |B| < \alpha, \\ \lambda\{B(t)+\alpha\} & \text{for } B < -\alpha, \end{cases} \tag{5.2}$$

where
$$F(t) = \frac{2K}{\pi}\int_0^\infty \frac{k\sin\omega t}{\omega}\,\mathrm{d}k, \tag{5.3}$$

$$H(t) = \frac{\sigma}{1+3K}J_2\sin\sigma t - \int_0^\infty \frac{\omega\sin\omega t\,\mathrm{e}^{-kd}}{\omega^2-\sigma^2}\,\mathrm{d}k, \tag{5.4}$$

and
$$\omega^2 = k(1+Kk^2). \tag{5.5}$$

For large t, the asymptotic value of H can be found by standard methods, and we find that

$$H(t) \sim \frac{\sigma(\sin\sigma t\,J_2 - \pi\mathrm{e}^{-d}\cos\sigma t)}{1+3K} + O(t^{-3}), \tag{5.6}$$

where J_2 has already been defined in (4.10). Since the transient behaviour of the solution started from rest is of no interest, and we require the ultimate periodic behaviour, we can replace $H(t)$ in (5.2) with its asymptotic value. The integral equation (5.2) is identical with that solved in Hocking (1987*b*) for the vertically oscillating plate, except for the presence of the term involving $H(t)$, which thus represents the contribution made by the source to the solution. Since the integral equation has an evolutionary character, the value of $B(t)$ can be determined from its values at preceding time-steps. More details of the numerical scheme can be found in Hocking (1987*b*).

The amplitude of the outgoing wave and the values of the radiated and dissipated energies over one period can be calculated when the periodic solution has been attained. The values of these quantities are given by the same equations as before, except that the quantities B_r and B_i in (4.12)–(4.14) are now given by the equations

$$B_r = \sigma\pi^{-1}\int B\cos\sigma t\,\mathrm{d}t, \quad B_i = \sigma\pi^{-1}\int B\sin\sigma t\,\mathrm{d}t, \tag{5.7}$$

where the integrals are taken over one period.

Since there are now five parameters in the problem, the results to be presented are limited to the special case in which $d = b = 1$. Figure 5 shows how the fraction of energy dissipated varies with the hysteresis angle, for three values of λ and for three values of K. As α is increased, the amount of energy dissipated decreases until it disappears at a critical value of α. The contact line then remains fixed relative to the plate with no energy dissipation. The motion in this case has already been given in §4, since setting $\lambda = 0$ there is equivalent to fixing the position of the contact line. Figure 5 shows that there is a minor exception to the monotonic decrease of E_D/E_S; in some cases, there is an initial increase in dissipation for small values of α. This increase was also encountered by Young & Davis (1987) and Hocking (1987*b*) in their studies of the vertically oscillating plate. It is probably produced by a phase shift between the two factors in the energy-dissipation integral (2.11).

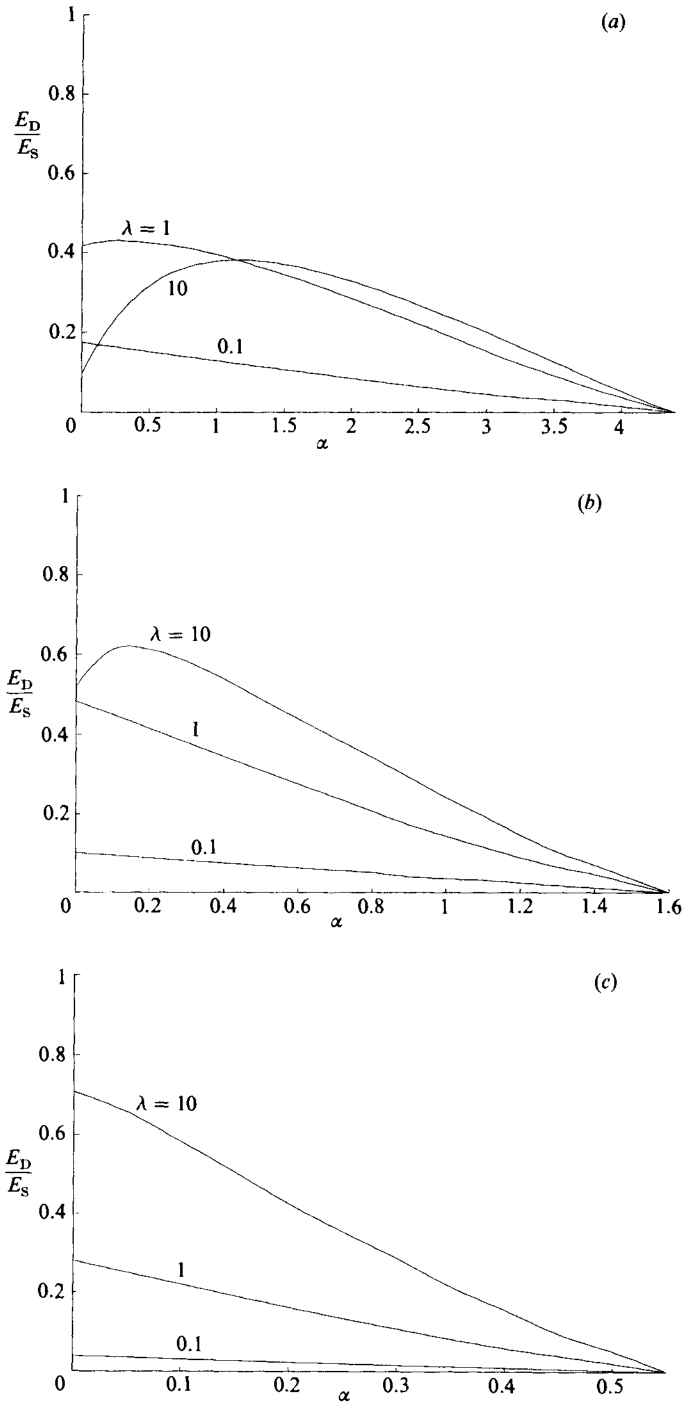

FIGURE 5. Dissipated energy as a fraction of the input energy for a source and plate, with hysteresis: $b = d = 1$. (a) $K = 0.1$, (b) 1.0, (c) 10.0.

6. Conclusions

One conclusion to be drawn from the results presented here is that the source-and-plate model provides an adequate replacement for a general body in the study of its heaving motion. Although not quantitatively accurate, it enables the main trends in the parameter dependence of the motion to be identified, and it probably provides at least order-of-magnitude estimates for such quantities as the amplitude of the radiated wave and the energy dissipation.

The condition applied at the contact line has a great influence on the amplitude of the radiated capillary–gravity wave. For a fixed contact line, or large contact-angle hysteresis, there is no dissipation of energy and the forced motion of the contact line as the body moves up and down and the displacement of fluid by the body contribute comparable amounts to the wave amplitude. The edge effect dominate when the width of the body is small and when the source is very deep, since the displacement effect is then very much reduced. Even when the body is large, the wave owes a considerable part of its amplitude to the conditions at the edge. Large values of the non-dimensional quantity b imply that the wavelength is small compared with the body width, or, for a body of fixed size, that the heaving motion has a long period.

When the contact line can slip along the body surface ($\lambda \neq 0$), energy is dissipated at the edge. For the vertically oscillating plate, this dissipation can be as large as 90% of the energy input (Hocking 1987*b*). For the oscillating bodies studied here this dissipation is relatively less important. The displacement effect is largely unaffected by the conditions at the edge and energy has to be introduced into the system to maintain the motion. The presence of contact-angle hysteresis induces a general decrease in the amount of energy dissipated, with a complete elimination of dissipation via the contact line when the amount of hysteresis is sufficient to prevent the contact line moving relative to the body.

The preparation of this paper was completed during a visit to Northwestern University and was supported by a grant from the National Science Foundation, Fluid Mechanics Program, to Professor S. H. Davis, grant no. MSM-8309520.

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